Bernoulli Differential Equation

The growth rate in sales of a new product that enters the market is given by the equation

$$y' = ry(M - y)$$

Where y is a function of time t. $r > 0$ is the proportionality constant and $M > 0$ is a constant that indicates the maximum sale that can be made according to production constraints.

1. Find the solution $y(t)$ of the equation.
2. Find the solution $y(t)$ of the given equation for $M = 10$ and knowing that $y(0) = 1$.

Solution

1. We rewrite the equation:
$$y' - ryM = -ry^2$$
From here, it can be solved by separable variables or as Bernoulli. As separable variables, it leads to a somewhat complicated expression, so we continue operating as if it were a Bernoulli. Then we divide everything by y^2:
$$\frac{y'}{y^2} - ry^{-1}M = -r$$
We make a substitution: $z = y^{-1}$ and $z' = -y^{-2}y'$
$$-z' - rzM = -r$$
$$z' + rzM - r = 0$$
We propose the following substitution: $z = uv$, so $z' = u'v + v'u$. Replacing:
$$u'v + v'u + ruvM - r = 0$$
$$v(u' + ruM) + v'u = r$$
This can be posed as a system where: $v(u' + ruM) = 0$ and $v'u = r$. We solve the first equation:
$$u' = -ruM$$
$$\frac{du}{dt} = -ruM$$
$$\frac{du}{u} = -rMdt$$
$$ln(u) = -rtM$$
$$u = e^{-rtM}$$
Replacing in the second equation:
$$v'(e^{-rtM}) = r$$
$$\frac{dv}{dt}(e^{-rtM}) = r$$
$$dv = e^{rtM}rdt$$
Then integrating both sides:
$$v = e^{rtM}\frac{r}{rM} + C$$
$$v = e^{rtM}\frac{1}{M} + C$$
With this we find our variable $z = uv$
$$z = e^{-rtM}(e^{rtM}\frac{1}{M} + C)$$
$$z = (\frac{1}{M} + Ce^{-rtM})$$
We replace to obtain $y = z^{-1}$
$$y = (\frac{1}{M} + Ce^{-rtM})^{-1}$$

2. If $M = 10$ and we evaluate at the point (0,1):

$$1 = (\frac{1}{10} + Ce^{0})^{-1}$$

$$1 = (1/10 + C)^{-1}$$

$$1 - 1/10 = C$$

$$C = 9/10$$

Then:

$$y = (\frac{1}{10} + \frac{9}{10}e^{-rt10})^{-1}$$